PHYSICAL MODEL AND EQUATION OF MOTION FOR A TWO-PHASE MEDIUM TAKING INTO ACCOUNT RANDOM PARTICLE (BUBBLES) MOTION

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Two-phase flows of incompressible fluids with solid particles or bubbles are examined. The source for random motion of a large number of particles moving relative to the fluid is found. New dimensionless parameters that affect the interaction of phases and the intensity of the random particle motion are found. Asymptotically exact expressions are obtained for the phase pressures and the interaction forces taking into account the gradient terms. The conditions for applicability of the diffusion approximation in the hydrodynamics of two-phase media are found.

Random motion in dispersed media has been studied by the methods of kinetic theory. Two-continuum equations of motion of a rarefied dispersed and carrier medium were obtained in [1]. A statistical approach to dispersed systems was developed in [2] assuming that the change in the velocity of particles between two collisions is small compared to the averaged value of the velocities of the random motion. In what follows, we study the opposite limiting case. The experiments in [3, 4] on the motion of bubble systems, from which it is evident that random motion of bubbles can appear due to their hydrodynamic interaction, are of fundamental significance in understanding this case. It is interesting to apply the theory of similarity [5] and the results of exact averaging of the equations of mechanics in [6] to studying random motion in two-phase media.

1. Formulation of the Problem and the Method of Solution

We are examining a viscous incompressible fluid, containing a large number of nondeformable spherical solid particles or bubbles with radius R in a field of potential mass forces g. The volume concentration c is not small.

Let the following inequalities be satisfied:

$$L \gg R,\ T \gg R/w, \tag{1.1}$$

where L is the characteristic distance over which the average flow parameters change; T is the characteristic time over which they change; w is the relative velocity of the phases. Then, on the strength of the first inequality, it is possible to use the approach of a continuous medium.

The second inequality (1.1) indicates that over the characteristic time T the relative displacement of particles greatly exceeds their radius, and in this case, the relative motion of the phases is significant and it is necessary to use a two-continuum approach.

Let the accelerations of the phases be equal and constant. We will transform to a system moving with the average acceleration of the fluid $d\mathbf{v}/dt$. Then, the mass force equals $\mathbf{g} - d\mathbf{v}/dt$. Including this force in the pressure, we obtain an equivalent system without mass forces. In this case, the external forces

$$\mathbf{F}^{(\alpha)} = (\rho_s - \rho)(\mathbf{g} - d\mathbf{v}/dt)V \qquad (\alpha = 1, 2, \ldots), \tag{1.2}$$

which are the only source of relative motion of the phases, will act on the particles in a volume V and with density ρ_s.

Assume now that the acceleration of the dispersed phase $d\mathbf{u}/dt$ is not equal to the acceleration of the fluid, but their difference is everywhere small

$$|d\mathbf{u}/dt - d\mathbf{v}/dt| \ll |\mathbf{g} - d\mathbf{v}/dt|. \tag{1.3}$$

Under condition (1.3), we assume in the leading approximation that the accelerations of the phases are equal and, assuming that $R \ll L$, we can neglect the gradients of the average

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quantities. Then they can be taken into account as corrections in the next approximation. This work is based on this method and inequalities (1.1) and (1.3) are, in this case, the only assumptions.

It can be shown (see Section 11) that condition (1.3) is valid if, in addition to (1.1), the following conditions are satisfied:

$$\mathrm{Fr}^2 R/wT \ll 1,\ \mathrm{Fr}^2 R/L \ll 1; \tag{1.4}$$

$$\mathrm{Fr}^2 = w^2/Rg',\ g' = |\mathbf{g} - d\mathbf{v}/dt|, \tag{1.5}$$

where Fr is Froude's number. Conditions (1.4) are not significant limitations for a concentrated two-phase medium.*

2. Random Motion of Particles. Dimensionless Criteria

Let us examine the motion of a system of particles in a fluid at the microscopic level. Mathematically, stationary states of this system are possible when the particles move along nodes of a regular periodic lattice under the action of identical forces $\mathbf{F}^{(\alpha)}$ (1.2) and the fluid simply percolates through a fixed structure. There arises the fundamental problem of the stability of such states.

It is well known [7] that an ordered system of particles moving in an ideal fluid, due to the interaction of these particles, is unstable. Small perturbations of the particle coordinates increase exponentially with time.

For small Reynolds numbers the stationary state of the system of particles, moving in the fluid under constant external forces, also turns out to be exponentially unstable. The characteristic time for breakdown of the stationary state is $\tau \sim R/w$, if the distance between particles is $\sim R$, i.e., essentially, ordered systems cannot exist. This is confirmed by all experiments [3, 4, 8, 9].

The instability of stationary states of motion of a system of particles relative to the fluid suggests that this system has stochastic properties and its behavior does not depend on the details of the initial particle distribution on the microscopic level.

Such a physical model, following from an analysis of the stability of stationary states, agrees with the understanding of random motion of separate bubbles within a bubbling bed with finite Reynolds numbers ($\mathrm{Re} = Rw/\nu$) [3, 4].

It is interesting to apply the theory of similarity [5] to the random motion of many particles in a fluid under the action of external forces $\mathbf{F}^{(\alpha)}$ (1.2).

Assume that Reynolds number is small ($\mathrm{Re} \ll 1$) and Stokes' equations are valid. Then the change in the particle coordinates $x^{(\alpha)}$ and their angular velocities $\omega^{(\alpha)}$ is determined by the action of the external forces (1.2) and hydrodynamic forces and moments, linear with respect to the velocities:

$$\rho_s V \ddot{x}_i^{(\alpha)} = V(\rho_s - \rho) g_i' - \mu R \sum_\beta \left(\varphi_{ij}^{(\alpha\beta)} \dot{x}_j^{(\beta)} + \psi_{ij}^{(\alpha\beta)} R \omega_j^{(\beta)}\right) - \partial U/\partial x_i^{(\alpha)}, \tag{2.1}$$

$$\frac{2}{5}\rho_s R^2 V \dot{\omega}_i^{(\alpha)} = R^2 \mu \sum_\beta \left(\varkappa_{ij}^{(\alpha\beta)} \dot{x}_j^{(\beta)} + R \chi_{ij}^{(\alpha\beta)} \omega_j^{(\beta)}\right),\ V = \frac{4}{3}\pi R^3, \ldots \mathbf{g}' = \mathbf{g} - d\mathbf{v}/dt.$$

The dimensionless functions $\varphi^{(\alpha\beta)}, \psi^{(\alpha\beta)}, \varkappa^{(\alpha\beta)}$, and $\chi^{(\alpha\beta)}$ depend only on the dimensionless coordinates of all particles relative to a fixed coordinate $(x^{(1)} - x^{(\alpha)})/R$, $(x^{(2)} - x^{(\alpha)})/R$, ...

Equations (2.1) are written in a system of coordinates in which the velocity of the fluid (averaged over the volume of a sphere with quite large radius) vanishes $\mathbf{v} = 0$. The quantity U on the right side of the first equation in (2.1) describes the interaction of particles on impact; $U = \infty$ when the distance between the surfaces of any particles becomes zero. It is correct to take into account the inertial terms in (2.1) only for $\rho_s \gg \rho$. In the opposite case, to within small terms of order $\sim\mathrm{Re}$, they should be omitted. In the case of comparable densities $\rho_s \leqslant \rho$, solving the linear system (2.1) relative to the velocities

*From the theory developed in what follows, it follows that conditions (1.4) are always satisfied if the concentrated two-phase medium is stable. They break down only in the case of strong instability.

of all particles, we will determine the velocities of particles as a function of their relative positions:

$$\frac{dx_i^{(\alpha)}}{dt} = \frac{(\rho_s - \rho) R^2}{\mu} g_j' F_{ij}^{(\alpha)} \left(\frac{x^{(1)} - x^{(\alpha)}}{R}, \frac{x^{(2)} - x^{(\alpha)}}{R}, \ldots \right), \tag{2.2}$$

where i, j = 1, 2, 3; α = 1, 2, ...; $F_{ij}^{(\alpha)}$ is a dimensionless function. The random velocity of the particles with Re $\ll$ 1 is related according to (2.2) to the random position of neighboring particles next to the given particle.

The random process of change of the coordinates, described by (2.2), for different values of the particle radius R and of the product $(\rho_s - \rho)R^2 g'/\mu$ is reduced, by a similarity transformation, to a single random process. It is important that in this case all random characteristics of the mutual position of particles turn out to be similar and depend only on the geometrical parameters and concentration c. It is important that there is no isotropy, since a direction w has been singled out. A single similarity criterion follows from Eqs. (2.1) (aside from the parameter c):

$$\Pi = \rho_s |\rho_s - \rho| |g'| R^3/\mu^2. \tag{2.3}$$

If Eqs. (2.1) are put into dimensional form, then the Π number will be the coefficient in front of the inertial term d^2x/dt^2. Thus, Π determines the contribution of inertial terms in Eqs. (2.1), which becomes important for sufficiently large number Π.

Similarity Criterion for Finite Reynolds Number. Without writing out the equations, it is possible to indicate the determining parameters of the system ρ_s, $|\rho_s - \rho| g'$, ρ, μ, R, and c. From these quantities, it is possible to form three dimensionless criteria: Archimedes' number Ar, the ratio of densities χ, and concentration c:

$$\text{Ar} = \rho |\rho_s - \rho| |g'| R^3/\mu^2, \ \chi = \rho_s/\rho. \tag{2.4}$$

It is fundamentally important to take into account two similarity criteria, since, in particular, for Re $\ll$ 1, neither Ar nor χ separately is a similarity criterion. Instead of (2.4), the criterion will be $\Pi = \chi$ Ar. The appearance of the number χ or Π is related to the random nature of the two-phase medium. Until now these numbers were not taken into account, since the relative motion of the phases was viewed as percolation of the fluid through a fixed particle structure.

3. Equation for Average Quantities. Froude's Number

Let Re $\ll$ 1, then according to (2.1) and (2.3) and based on the theory of similarity [5], the relative velocity of the phases is

$$\mathbf{w} = \mathbf{u} - \mathbf{v} = (\rho_s - \rho) g'/A, \ A = (\mu/R^2) G(c, \Pi), \tag{3.1}$$

where G is a dimensionless function. Let us introduce Froude's number (1.5).

Taking into account (3.1), the friction force $\mathbf{F}^*$, acting on a unit volume of the dis- dispersed phase, for Re $\ll$ 1, can be represented in the form

$$-\mathbf{F}^* = \mu R^{-2} G(c, \text{Fr}) \mathbf{w} = A \mathbf{w}, \tag{3.2}$$

since the resistance force of a separate particle on the average equals the motive force $\mathbf{F}^{(\alpha)}$ (1.2). If Eqs. (2.1) are put into dimensionless form, using the value of $\mathbf{w}$ (3.1), then the number $\text{Fr}^2 \approx \Pi G^2$ will occur in front of the inertial term in (2.1). It is evident that the simplified equations (2.2) correspond to the values Fr $\ll$ 1, while Eqs. (2.1) correspond to the values Fr $\geqslant$ 1.

It is evident from (3.2) that even for small Reynolds numbers the friction force for the phases can depend nonlinearly on the relative velocity of the phases w due to the effect of the number Fr on the chaotic particle motion. For finite Reynolds numbers, the friction force between the phases, from considerations of similarity, is written in the form

$$F^* = -C_w R^{-1} \rho |w| w, \ C_w = C_w(\text{Re}, \chi, c). \tag{3.3}$$

It is significant that the dimensionless coefficient C_W in (3.3) depends on the relative densities χ.

Numerous experiments on settling of solid particles in a fluid [10], whose conditions correspond to the values $Fr \ll 1$, give dependences of the form

$$u = u_0(1-c)^n,$$

where $n \approx 5$ for $Re \ll 1$. In these experiments, closed vessels were used. For this reason $(1-c)v = -cu$, i.e., there was a rising flux of fluid $v \neq 0$. Taking this into account, based on the data in [10], we obtain the following empirical dependence:

$$w = w_0(1-c)^{n-1},\quad G(c) = G_0(1-c)^{1-n},$$

where G_0 is a constant for $Re \ll 1$, $Fr \ll 1$. As the Reynolds number increases, the index n decreases from 5 to 2.5 [9, 10]. The effect of Froude's number Fr or the number χ on the settling rate has not yet been studied experimentally.

4. Local Nature of Generation of Chaotic Motion

Random motion is caused in the particle velocities by the presence of the motive forces $F^{(\alpha)}$ (1.2). An exchange of kinetic energy of particles between two macroscopic volumes is impossible. As a result of this, the kinetic energy is not a source of randomness. In order to verify this, let us estimate the characteristic distance λ_* over which the energy of translational particle motion with characteristic velocity w dissipates.

In the case $\rho_s \gtrsim \rho$, comparing the characteristic work of friction forces $\lambda_* F^* V \approx \lambda_* \rho_s g' V$ and kinetic energy $\sim \rho_s w^2 V$, we find taking into account (3.1) and (3.2) the estimate

$$\lambda_* \sim R\,\mathrm{Fr}^2. \tag{4.1}$$

The estimate (4.1) is also valid for the case $\rho_s \ll \rho$. In the general case, (4.1) can be represented in terms of the drag coefficient C_w in the form

$$\lambda_* \sim \frac{\rho_s}{\rho C_w} R,\quad \rho_s \gtrsim \rho;\quad \lambda_* \sim \frac{R}{C_w},\quad \rho_s \ll \rho. \tag{4.2}$$

The estimates (4.1) and (4.2) show that $\lambda_* \gg R$ only for large Froude number ($Fr \gg 1$) or very heavy particles ($\rho_s/\rho \gg 1$).

If, in addition to estimates (4.1) and (4.2), the restriction (1.3) is taken into account, which is necessary to impose on the characteristic macroscopic scale L, then we obtain that $\lambda_* \ll L$. From this follows an important result. Exchange of kinetic energy of chaotic motion between regions at macroscopic distance $L \gg R$ is impossible. At each point of the two-phase continuous medium, a local equilibrium state is established, independent of the neighboring macroscopic regions. The mechanism of heat transfer, which is characteristic of an inhomogeneous rarefied gas, is absent.

5. Estimate of the Intensity of Random Particle Motion

Let us try to obtain an energy estimate for the intensity of chaos, starting from the ideas of Sec. 2. First, we note that in a random system of particles, moving in a fluid, fluctuations on different scales are present. In this case, velocity fluctuations are correlated only for quite close particles, situated at distances $\sim R$. Therefore, the scale of fluctuations is $\sim R$.

For small Reynolds and Froude's numbers, Eqs. (2.2) are correct and from them it follows that in order of magnitude

$$\sqrt{\overline{(\delta u)^2}} = |w|\,\varphi(c), \tag{5.1}$$

where $\varphi(c) \sim 1$ in a concentrated system. In a rarefied system, the perturbation of particle velocity caused by the presence of neighboring particles is of the order of $c^{1/3}$, so that for $c \to 0$, $\varphi \sim c^{1/3}$.

In the general case, the change in the kinetic energy of particles and fluid in fluctuation motion on the scale $\sim R$ is limited by the work of the external force $F^{(\alpha)}$ (1.2) on this scale. Part of this work over the lifetime of the fluctuation $\tau \sim R/\delta u$ is dissipated due to viscous friction. To within a function of concentration c, we have

$$\rho_s(\delta u)^2 \lesssim |\rho_s - \rho|\, g' R,\quad g' = |\mathbf{g} - d\mathbf{v}/dt|. \tag{5.2}$$

If $\rho_s \ll \rho$, then on the left side of (5.2) we will have $\rho(\delta u)^2$. For Fr $\gtrsim$ 1, it follows from (5.2) that

$$\sqrt{\overline{(\delta u)^2}} \simeq |w| \varphi_1(c) \mathrm{Fr}^{-1}. \tag{5.3}$$

For finite Reynolds numbers and for density ratios $\rho_s \gtrsim \rho$, it follows from (5.2), (1.2), and (3.3) that

$$\sqrt{\overline{(\delta u)^2}} \lesssim w \sqrt{C_w \rho/\rho_s}, \quad \mathrm{Re} > 1. \tag{5.4}$$

In a concentrated system, $C_w \gtrsim 1$.

It follows from Eqs. (5.3) and (5.4) that the intensity of random motion decreases with increasing particle density. Velocity fluctuations δu exist for Re $\ll$ 1 only if Fr $\lesssim$ 1. In the region Fr$\to\infty$, the quantity $\delta u/w \to 0$, and there are practically no velocity fluctuations. Similarly, for finite Reynolds numbers, for heavy particles ($\rho_s/\rho \gg 1$), the amplitude of fluctuations $\delta u \ll w$.

The conditions obtained for the existence of randomness in velocities are much more restrictive than the conditions of validity (1.3) or the diffusion approximation (see Sec. 11).

It is also evident by comparing (5.3) and (5.4) with (4.1) and (4.2) that in the region of parameters where there is significant randomness in velocities exchange of kinetic energy of fluctuating motion between macroscopic regions is always absent.

6. Equations of Motion

Let us proceed to determine the small corrections to the equations of the leading approximation in Sec. 3. We will write the averaged equations of mechanics in exact form [6]:

$$(1-c)\rho d\mathbf{v}/dt + c\rho_s d\mathbf{u}/dt = (1-c)\rho\mathbf{g} + c\rho_s\mathbf{g} - \nabla p + \mathrm{div}\,[-(1-c)\rho\overline{\delta\mathbf{v}\delta\mathbf{v}} - c\rho_s\overline{\delta\mathbf{u}\delta\mathbf{u}} + \sigma], \tag{6.1}$$
$$c\rho_s d\mathbf{u}/dt = c\rho_s\mathbf{g} - \mathrm{div}(c\rho_s\,\overline{\delta\mathbf{u}\delta\mathbf{u}}) + c\mathbf{F},$$
$$\partial c/\partial t + \mathrm{div}\,(c\mathbf{u}) = 0, \ -\partial c/\partial t + \mathrm{div}\,[(1-c)\mathbf{v}] = 0.$$

The first equation in (6.1) describes the motion of the mixture as a whole; σ is the viscous stress; p is the pressure. The bar indicates averages with respect to fluctuations. In the second equation in (6.1), describing particle motion, $\mathbf{F}$ is the average force acting on a given particle from the side of the flux and, generally speaking, other particles. $\mathbf{F}$, in particular, includes viscous stresses in the medium of particles, which are second-order infinitesimals with respect to $R/L \ll 1$, i.e., they are negligibly small.

In the leading approximation with respect to the small parameters (1.1), the small gradients should be neglected in Eqs. (6.1) and $d\mathbf{u}/dt = d\mathbf{v}dt$ should be used (Sec. 1). In this case, the phase interaction force $\mathbf{F}_{(0)}$ is determined according to Sec. 3 in the form

$$\mathbf{F}_{(0)} = -\rho\mathbf{g}' + \mathbf{F}^*, \tag{6.2}$$

where $\mathbf{F}^*$ is defined by Eq. (3.2) or (3.3).

In the next approximation, in Eqs. (5.1), it is necessary to take into account small corrections which are linear with respect to the gradients of average quantities and the mass force $\mathbf{g}'$:

$$\mathbf{F} = \mathbf{F}_{(0)} + \mathbf{k}_1\nabla c + \mathbf{k}_2\nabla\mathbf{g}' + \mathbf{k}_3\nabla\mathbf{v} + \mathbf{k}_4 d\mathbf{g}'/dt. \tag{6.3}$$

The quantities $\nabla\mathbf{u}$ and $\nabla\mathbf{w}$ did not enter into (6.3), since according to Eqs. (3.1) and (3.3) they are expressed in terms of ∇c and $\nabla\mathbf{v}$. For a similar reason, dc/dt is not taken into account in (6.3).

The tensor coefficients $\mathbf{k}_a$ have the form

$$\mathbf{k}_a = Q_a\varphi_a'(c, \mathrm{Ar}, \chi, \mathbf{e}) = Q_a\varphi_a(c, \mathrm{Re}, \chi, \mathbf{e}), \tag{6.4}$$

where Q_a are scalar dimensional combinations of the parameters R, μ, ρ; φ_a are dimensionless tensor functions; $\mathbf{e}$ is a unit vector along $\mathbf{w}$. Similarly, the pressure tensor in (6.1), due to pressure arising as a result of fluctuations in particle velocities, equals

$$\mathbf{P}_s = c\rho_s\overline{\delta\mathbf{u}\delta\mathbf{u}} = \rho_s w^2\mathbf{k}_5(c, \mathrm{Re}, \chi, \mathbf{e}). \quad (6.5)$$

The average quantity $\overline{\delta\mathbf{v}\delta\mathbf{v}}$ is calculated in the same form. Equations (6.1)-(6.4) together with (3.1)-(3.3) represent the equations of a two-phase medium taking into account gradients. The form of the unknown tensor functions $\mathbf{k}_a$ can be found with the help of the theory of similarity and symmetry considerations [11].

7. General Equation for the Force F for Re ≪ 1

In determining the dependence (6.2), it is necessary to take into account that the space is isotropic and the tensor functions $\mathbf{k}_a$ depend on the direction of the vector $\mathbf{g}'$ or $\mathbf{w}$.

From Eqs. (6.2)-(6.4) and (3.1), (3.2), taking into account similarity and symmetry, we find

$$\mathbf{F} = -\rho\mathbf{g}' - A(\mathbf{w} + \mathbf{a}\nabla c + \mathbf{b}\nabla\mathbf{v} + \mathbf{d}\nabla\mathbf{g}' + \mathbf{h}d\mathbf{g}'/dt), \quad (7.1)$$

where the tensor coefficients are defined in terms of $e_i = w_i/|\mathbf{w}|$ by the equations

$$a_{ij} = R|\mathbf{w}|(f_1\delta_{ij} + f_2e_ie_j), \quad (7.2)$$
$$b_{ijk} = R(b_1e_i\delta_{jk} + b_2e_j\delta_{ik} + b_3e_k\delta_{ij} + b_4e_ie_je_k).$$

The tensors h_{ij} and d_{ijk} are written similarly to (7.2). In many practically important problems, it is possible to neglect a change in acceleration g' and, correspondingly, to drop the terms with **h** and **d** in (7.1). The scalar coefficients entering into (7.2) depend only on the concentrations c and Froude's number Fr. The coefficient $b_1 = 0$ in (7.2) without loss of generality, since in accordance with the equations of continuity (6.1) and taking into account the leading approximation (3.1), $\mathrm{div}\,\mathbf{v}$ is expressed in terms of ∇c and $\nabla g'$. The term $\mathbf{b}\nabla\mathbf{v}$ in (7.1), according to (7.2), is defined by the equation

$$|\mathbf{w}|R^{-1}(\mathbf{b}\nabla\mathbf{v})_i = b_2[\mathbf{w}, \mathrm{rot}\,\mathbf{v}]_i + (b_3 + b_2)(\mathbf{w}\nabla)v_i + b_4(\mathbf{e}(\mathbf{e}\nabla)\mathbf{v})w_i. \quad (7.3)$$

The first term on the right side of (7.3) determines the transverse force δF_1 acting on the dispersed phase. The force δF_1 differs from the well-known Magnus force by a coefficient. For Re ≪ 1, the quantity δF_1 exceeds the Magnus force by the factor Re^{-1} ($b_2 \neq 0$). The appearance of a new transverse force is related to taking into account the random particle motion. It is explained by the breakdown of the equilibrium state of random motion, introduced by a change in the velocity field over a long time of the order of $\tau \sim 1/|\nabla\mathbf{v}|$. It is clear that in this case there must indeed arise a correction to the force $\mathbf{F}^*$ of the order of $\tau/t \sim |\nabla\mathbf{v}|R/w$, where $\tau \sim R/w$ is a short characteristic time for establishing randomness in a concentrated system.

The transverse force $\delta\mathbf{F}_1$ has meaning for describing the motion of concentrated suspensions in the presence of shearing deformations and, in particular, for motion in vertical channels. It can cause considerable change in the concentration distribution over the channel cross section under conditions when the usual Magnus force equals zero.

Let us proceed to corrections in the force of interaction between phases (7.1), owing to the concentration gradient ∇c.

8. Diffusion Model of Forces (Re ≪ 1)

The reason for the appearance of the term $\mathbf{a}\nabla c$ in (7.1) is clear from the following model.

Let us separate the volume particle flux $\mathbf{J} = c\mathbf{u}$ into two parts: the hydrodynamic (systematic) flux $c\mathbf{u}_H$ and a diffusion flux $\mathbf{D}\nabla c$, stemming from chaotic motion of particles with a tensor diffusion coefficient D:

$$\mathbf{J} = c\mathbf{u} = c\mathbf{u}_H - \mathbf{D}\nabla c. \quad (8.1)$$

Assuming that the force of friction between the phases $\mathbf{F}^*$ depends on the systematic velocity u_H exactly as in the uniform state, we obtain from (8.1) and the equations of the leading approximation (3.2)

$$\mathbf{F}^* = \mathbf{F}^*_{(0)}(\mathbf{u}_H - \mathbf{v}) = -A\left(\mathbf{w} + \frac{1}{cA}\frac{\partial Aw}{\partial w}\mathbf{D}\nabla c\right). \quad (8.2)$$

Since the parameters on which the chaotic particle motion mainly depends are known, from the theory of similarity and symmetry considerations, we have

$$D_{ij} = D_\perp \delta_{ij} + (D - D_\perp) w_i w_j / w^2,$$
$$D = R|\mathbf{w}| f(c, \mathrm{Fr}), \quad D_\perp = R|\mathbf{w}| f_\perp(c, \mathrm{Fr}). \tag{8.3}$$

Expressions (8.2) and (8.3) are equivalent to taking into account the terms w and $a\nabla c$ in the friction force in (7.1). We note that transport coefficients of the form $D \sim Rw$ were introduced in [3, 4] in studying a system of floating bubbles with finite Reynolds numbers.

It is significant that diffusion is present for $\mathrm{Re}\to 0$ as well and, primarily, that the transport coefficient D greatly affects corrections of the order of R/L to the force of interaction between phases. Apparently, the term $\mathbf{D}\nabla c$, generally, is the leading correction.

The dependence of the diffusion coefficient D on the Froude number is important. Taking into account the fact that $D \sim R\delta u$, where δu is the mean-square fluctuation in the velocity, we obtain from (5.3) that $D\to 0$ for $\mathrm{Fr}\to\infty$, i.e., diffusion vanishes for large Froude numbers. Diffusion is noticeable for $\mathrm{Fr} < 1$.

9. Equations for Small Froude Numbers

In addition to the force F, Eqs. (6.1) include pressure fluctuations. The pressure tensor in a medium of particles $\mathbf{P}_s$ is determined from (6.5) in the form

$$(\mathbf{P}_s)_{ij} = \rho_s S_\perp w^2 \delta_{ij} + \rho_s (S - S_\perp) w_i w_j, \tag{9.1}$$

where the dimensionless functions S and $S_\perp$ depend on c and Fr. For small Reynolds numbers, a similar contribution $\overline{\delta\mathbf{v}\delta\mathbf{v}}$ to (6.1) is always negligibly small. It makes sense to take into account (9.1) for $\rho_s \gg \rho$.

For small concentration c, $\delta u \sim wc^{1/3}$, so that S, $S_\perp \sim c^{5/3}$.

For large Froude numbers ($\mathrm{Fr}\to\infty$), the functions S and $S_\perp$ in (9.1) decrease as Fr^{-2}. Therefore, the pressure $\mathbf{P}_s \to 0$ for $\mathrm{Fr}\to\infty$. For small Froude numbers ($\mathrm{Fr} \ll 1$), from (9.1) and (7.2), (3.2) follow estimates of the role of pressure $\mathbf{P}_s$ and corrections $\delta\mathbf{F}$ to the force $\mathbf{F}_{(0)}$:

$$\operatorname{div} \mathbf{P}_s \sim \rho_s w^2/L, \; \delta F \sim \mu w/RL \sim \rho w^2/\mathrm{Re}L.$$

Taking into account the fact that $\mathrm{Re}\,\rho_s/\rho \sim \mathrm{Fr}^2$, we obtain from here that for small Froude numbers the pressure of the dispersed phase is negligibly small, $\mathbf{P}_s \sim \mathrm{Fr}^2 \delta F$. It is evident from here that the pressure from particles $\mathbf{P}_s$ plays a secondary role in the equations compared to the diffusion terms, determined by (8.2) or (7.1).

Estimates show that in equations with $\mathrm{Fr} \ll 1$ the difference in the accelerated phases should be neglected, but it is necessary to take into account diffusion corrections to the force. In this case, the equation of relative motion of the phases follows from (6.1) and (7.1)

$$\mathbf{w} = (\rho - \rho_s)\mathbf{g}' A^{-1} - \mathbf{a}\nabla c - \mathbf{b}\nabla\mathbf{v}. \tag{9.2}$$

The simplified diffusion model of Sec. 8 corresponds to $\mathbf{b} = 0$ in Eq. (9.2).

10. Interphase Interaction and Pressure of Particles with Finite Reynolds Numbers

For Re > 1, the expression for the force following from (6.4), similar to (7.1), has the form

$$\mathbf{F} = -\rho\mathbf{g}' - R^{-1} C_w \rho |\mathbf{w}| [\mathbf{w} + \mathbf{a}\nabla c + \mathbf{b}\nabla\mathbf{v} + \mathbf{d}\nabla\mathbf{g}' + \mathbf{h}\, d\mathbf{g}'/dt]. \tag{10.1}$$

The coefficients **a**, **b**, **d**, **h** are again determined by equations of the type (7.2), but they now depend on $\chi = \rho_s/\rho$ and Re (aside from c). For $\mathrm{Re} \gg 1$, the coefficients in (10.1), evidently, depend weakly on Re. The coefficient a is similar to the tensor $c^{-1}D$ in the diffusion expression (8.2). The diffusion tensor D depends on the density ratio. On the strength of (5.4), the quantity $D \sim \sqrt{\rho/\rho_s}\to 0$ for $\chi\to\infty$. The diffusion term is significant only in the case of comparable particle and fluid densities $\rho_s \leqslant \rho$. The diffusion coefficients must be maximum in bubble media, studied in [3, 4]. For small changes in the acceleration of the flow, in (10.1), the coefficients d = 0 and h = 0.

The functions S and $S_{\perp}$ in the pressure $\mathbf{P}_s$ (9.1) depend on c, Re, and χ. However, the dependence on Re for Re $\gg$ 1, as usual, should be insignificant. But the dependence of S on χ is fundamental. Using (5.4), we see that $S\to 0$ for $\chi = \rho_s/\rho \to \infty$ as χ^{-1}. Thus, the pressure in the particle medium, stemming from particle collisions, approaches zero with increasing particle and fluid density ratio ρ_s/ρ.

APPENDIX

11. Condition of Applicability of the Approximation of Equal Accelerations

The momentum equation for the dispersed phase has the form [6]

$$\rho_s \frac{d\mathbf{u}}{dt} = \rho_s \mathbf{g} - \rho\left(\mathbf{g} - \frac{\partial \mathbf{v}}{\partial t}\right) - \frac{C_w}{R}\rho \mathbf{w}|\mathbf{w}|, \quad \mathbf{w} = \mathbf{u} - \mathbf{v}. \tag{11.1}$$

In the equations, we neglect small contributions of viscous and fluctuation stresses. Taking them into account does not essentially change the following estimates.

Expressing $\mathbf{u}\nabla\mathbf{u}$ in terms of $\mathbf{v}\nabla\mathbf{v}$, we transform (11.1) to the form

$$\rho_s\left(\frac{d\mathbf{v}}{dt} + \frac{\partial \mathbf{w}}{\partial t} + (\mathbf{v}\nabla)\mathbf{w} + (\mathbf{w}\nabla)\mathbf{v} + (\mathbf{w}\nabla)\mathbf{w} - \mathbf{g}\right) + \rho\left(\mathbf{g} - \frac{d\mathbf{v}}{dt}\right) = \frac{C_w}{R}\rho\mathbf{w}|\mathbf{w}|. \tag{11.2}$$

Let us introduce the dimensionless variables, corresponding to the space and time scales of variation of the flux parameters, L and T. Comparing terms containing w on the left and right sides of (11.2), we see that to within small quantities of the order of

$$\frac{R}{Tw}\frac{\rho_s}{\rho C_w} \ll 1, \quad \frac{R}{L}\frac{\rho_s}{\rho C_w} \ll 1 \tag{11.3}$$

terms with w on the left side of (11.2) can be neglected. The possibility of (11.3) stems from the starting inequalities (1.1). In this case, the equation of the diffusion approximation follows from (11.2)

$$(\rho - \rho_s)(\mathbf{g} - d\mathbf{v}/dt) = -R^{-1}C_w\rho\mathbf{w}|\mathbf{w}|. \tag{11.4}$$

The equation for the momenta of the two-phase medium transforms in a similar manner. In this case, the equations correspond to the motion of a mixture as a homogeneous mixture, while the velocity of relative motion of phases is determined by the acceleration $\mathbf{g}'$. This is the approximation of equal phase accelerations, when (1.3) is valid.

It is evident from (11.3) that for $C_w\rho/\rho_s \geqslant 1$ the conditions for the validity of the diffusion approximation are satisfied with the same accuracy as the conditions for applicability of the continuous medium approach (1.1). This is always satisfied for $\rho_s \leqslant \rho$, since $C_w \geqslant 1$. Therefore, the diffusion approximation is useful for bubbles, drops, or solid particles in a fluid.

Conditions (11.3) can break down only for heavy particles, when $\rho_s \gg \rho$ (e.g., particles in a gas) and if the scales L and T are not sufficiently large. In the case of small Reynolds numbers and $\rho_s \gg \rho$, conditions (11.3) can be rewritten in the form (1.4). The diffusion approximation may be inapplicable only for large Froude numbers Fr $\gg$ 1.

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FORMATION OF TAYLOR VORTICES BETWEEN HEATED ROTATING CYLINDERS

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Experimental observations [1-4] show that a secondary steady flow of the Taylor vortex type (rotationally symmetric toroidal vortex cells regularly positioned along the symmetry axis of the cylinders) can arise as the result of the loss of stability of a nonisothermal Couette flow between concentric cylinders rotating with different angular velocities. This secondary flow was found in [5] by the Lyapunov–Shmidt method in the case in which the cylinders are rotating in the same direction and the Prandtl number is equal to unity.

Results are presented in this paper of calculations of Taylor vortices both for the case in which the cylinders rotate in the same direction and for the case of an opposite rotation direction of the cylinders. The change in the structure of the vortices as the values of the parameters of the problem vary is illustrated by the pattern of the stream lines of the secondary flow. Analytic dependences of the amplitude of the vortices and the decrement of a nonisothermal Couette flow on the Prandtl number are obtained, which eliminate the need to make time-consuming calculations and permit establishing some properties of the fundamental and secondary regimes. One should note that a similar dependence of the amplitude of the secondary regime on the Prandtl number for the steady problem of free convection in a layer of liquid was established and used in the calculations in [6].

1. The Lyapunov–Shmidt Series. Let a viscous uniform heat-conducting liquid fill the cavity between two infinite solid concentric cylinders. The radii, angular velocities, and temperatures of the inner and outer cylinders will be denoted by R_1, Ω_1, Θ_1 and R_2, Ω_2, Θ_2, respectively.

We will assume that there are no external body forces and the discharge rate of the liquid through the transverse cross section of the cavity of the cylinders is equal to zero. Then the Navier–Stokes equations and the thermal conductivity equation permit an exact solution (a nonisothermal Couette flow) with the velocity vector $\mathbf{U}_0 = \{u_{0r}, u_{0\varphi}, u_{0z}\}$, temperature T_0, and pressure Π_0 (r, φ, and z are dimensionless cylindrical coordinates):

$$\mathbf{U}_0 = \{0,\ V_0(r),\ 0\},\ V_0 = ar + b/r,\ T_0 = c\ln r + 1, \tag{1.1}$$
$$\Pi_0 = \int\limits_1^r \frac{V_0^2(\rho)}{\rho}\left(1 - \frac{\mu}{\Pr}\ln\rho\right)d\rho + \text{const},$$
$$a = (\Omega R^2 - 1)/(R^2 - 1),\ b = 1 - a,\ c = (\Theta - 1)/\ln R,$$

where $\mu = \beta c \Theta_1 \Pr$ is the Rayleigh number, $\Pr = \nu/\chi$ is the Prandtl number, β, ν, and χ are the thermal expansion, kinematic viscosity, and thermal conductivity coefficients, respectively, $R = R_2/R_1$, $\Omega = \Omega_2/\Omega_1$, and $\Theta = \Theta_2/\Theta_1$.

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